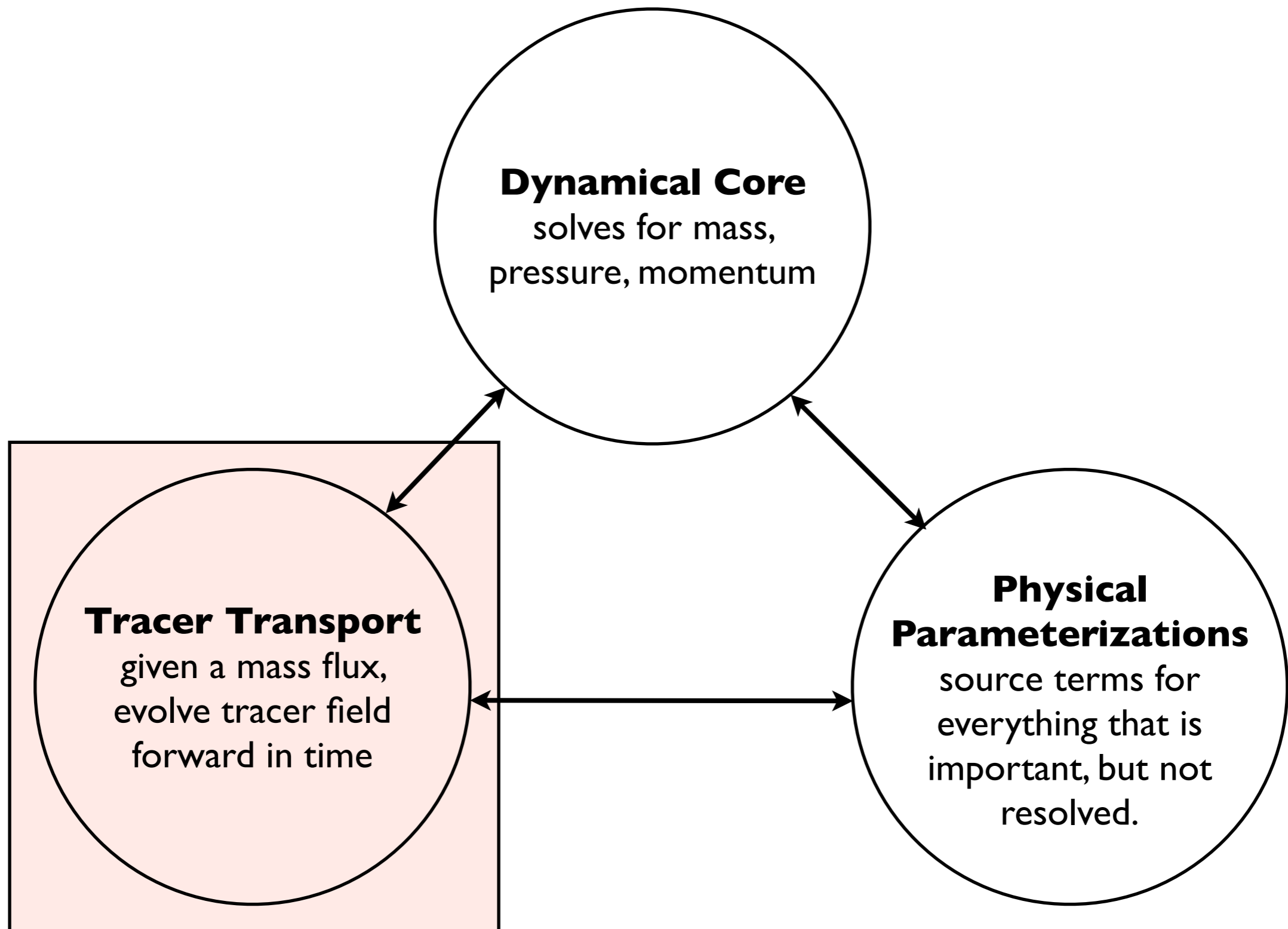


A High-Order Transport Scheme for Unstructured Atmosphere and Ocean Climate Models

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A Conceptual and Practical View of Atmosphere and Ocean Climate Model Components



Transport

$$\frac{D}{Dt} \equiv \partial_t + \vec{u} \cdot \nabla$$

advective form

$$\frac{DT}{Dt} = 0$$

flux form

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\partial_t (\rho T) + \nabla \cdot (\rho T \vec{u}) = 0$$



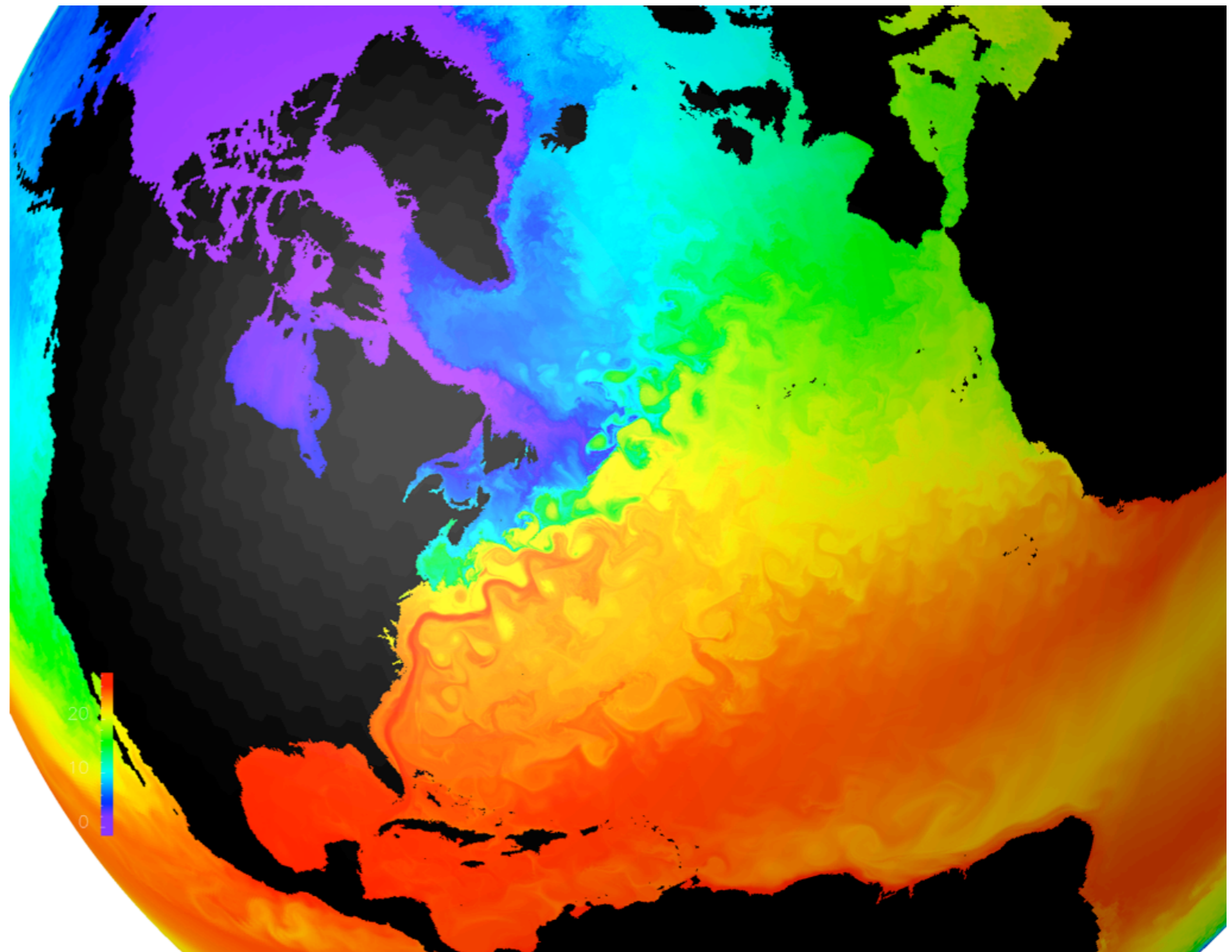
Note! Coupled to the mass variable.

Importance of Transport in the Ocean and Atmosphere

Transport is arguably the single most important process in the climate system.

The net excess of energy entering the climate system in the tropics is moved polewards in the ocean and atmosphere primarily through transport.

The transport of momentum by the turbulent ocean/atmosphere flows play an $O(1)$ role in determining mean currents/winds of the ocean/atmosphere.



The transport of trace-constituents (water vapor, aerosols, algae, plankton, etc) are essential for even the most basic description of the climate system.

What attributes do we seek in our transport schemes?

1. Locally conservative (and, thus, globally conservative) for some domain Ω .

$$\int_{d\Omega} \partial_t(\rho T) d\Omega = - \int_{d\Omega} \nabla \cdot (\rho T \vec{u}) d\Omega$$

2. Accurate in the sense that the error goes to zero as grid-spacing and time-step raised to some power (preferably greater than one!)

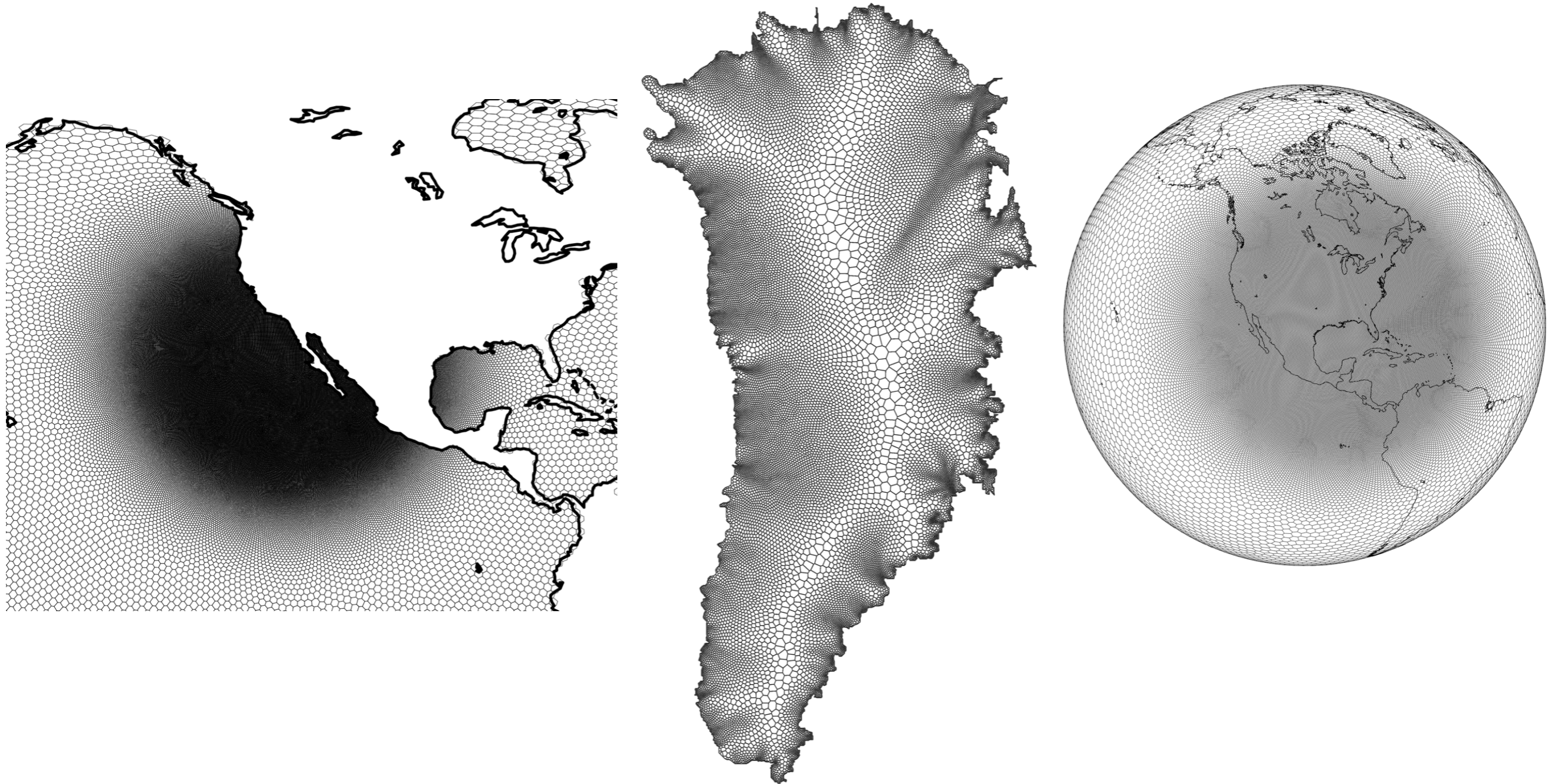
3. Computationally efficient. We are planning for $O(100)$ tracers. Computing climate solutions is a computational grand challenge. Within that challenge, transport is a significant “user” of FLOPs.

4. Monotone in the sense that the process of transport does not create new extrema.

$$\max [T(\vec{x} + \vec{r}, t^n)] \geq T(\vec{x}, t^{n+1}) \leq \min [T(\vec{x} + \vec{r}, t^n)]$$

These attributes are in conflict.

And we want all of these properties to hold on multi-scale meshes.
(i.e. meshes composed of arbitrary, convex polygons in 2D)



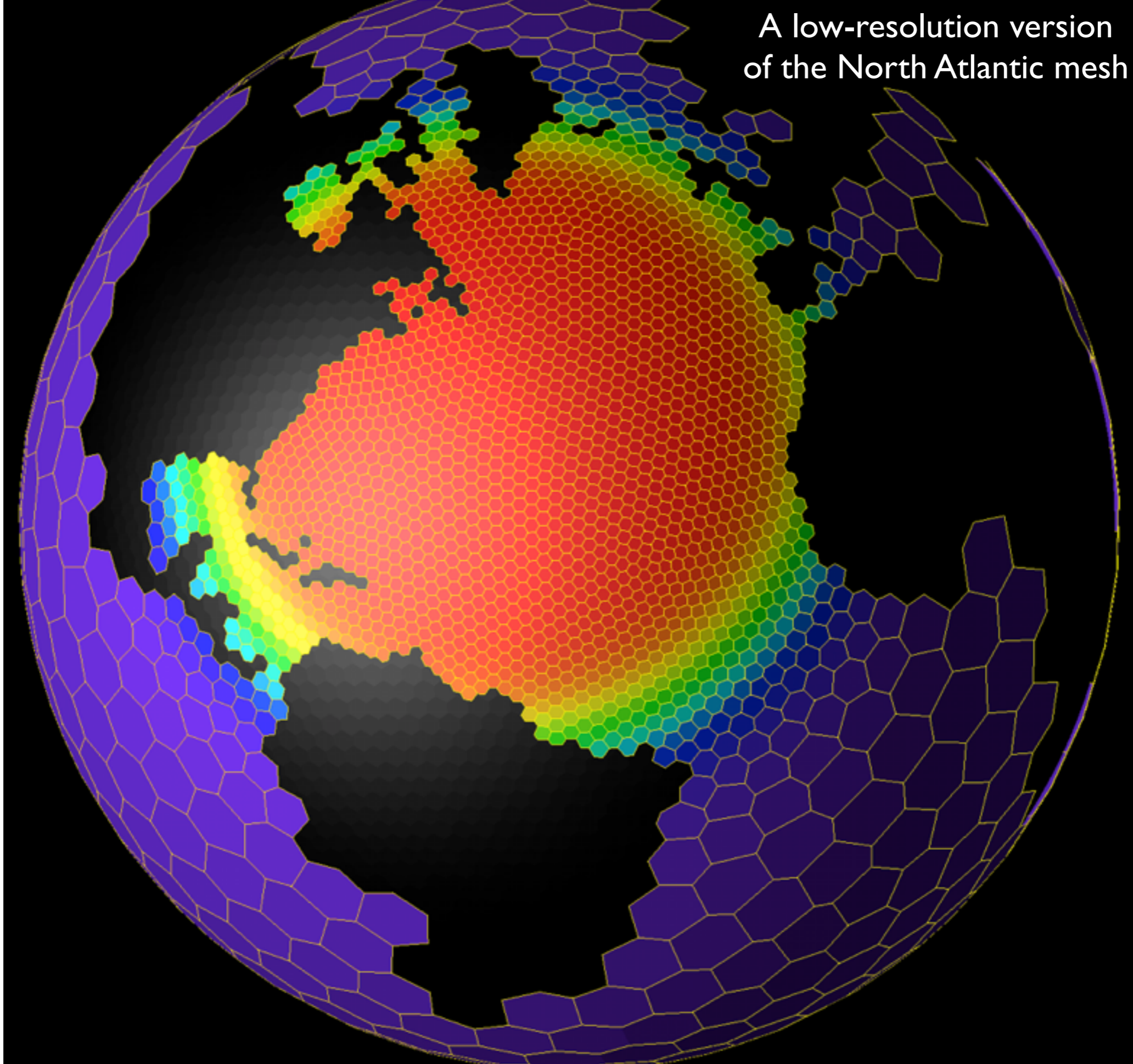
- Ju, L., T. Ringler and M. Gunzburger, 2009, Voronoi Tessellations and their Application to Climate and Global Modeling, Numerical Techniques for Global Atmospheric Models, Lecture Notes in Computational Science. ([pdf](#)).

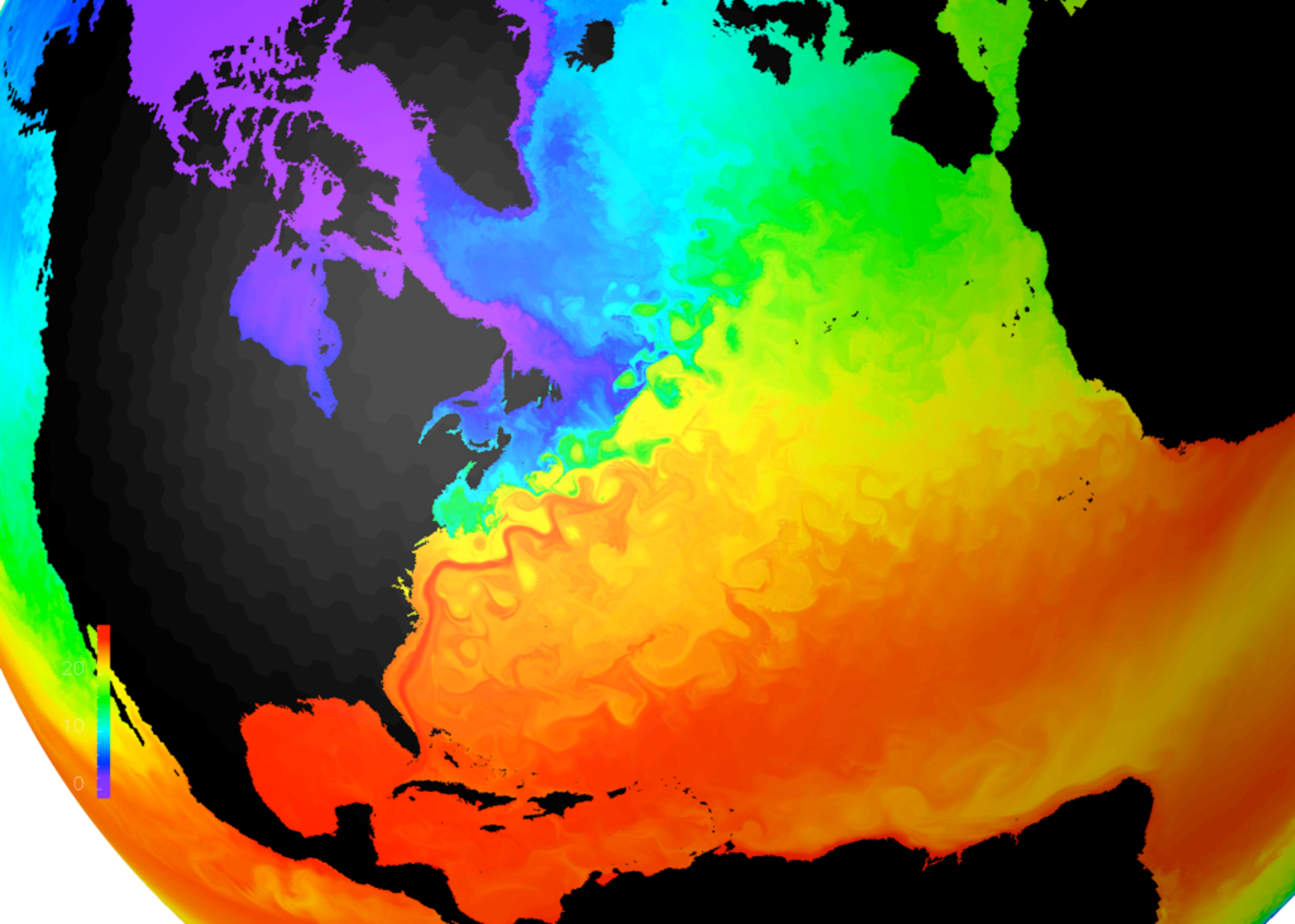
Context: Building a Global, Multi-Scale Climate System Model

1. MPAS is an unstructured-grid approach to climate system modeling.
2. MPAS supports both quasi-uniform and variable resolution meshing of the sphere using quadrilaterals, triangles or Voronoi tessellations.
3. MPAS is a software framework for the rapid prototyping of single-components of climate system models (atmosphere, ocean, land ice, etc.).
4. MPAS offers the potential to explore regional-scale climate change within the context of global climate system modeling. Multiple high-resolution regions are permitted.
5. MPAS is currently structured as a partnership between NCAR MMM and LANL COSIM, where we intend to distribute our models through open-source, 3rd-party facilities (e.g. Sourceforge).



A low-resolution version
of the North Atlantic mesh





20

10

0

OK, how should we do transport?

Alternative #1: Traditional Reconstruction with FCT

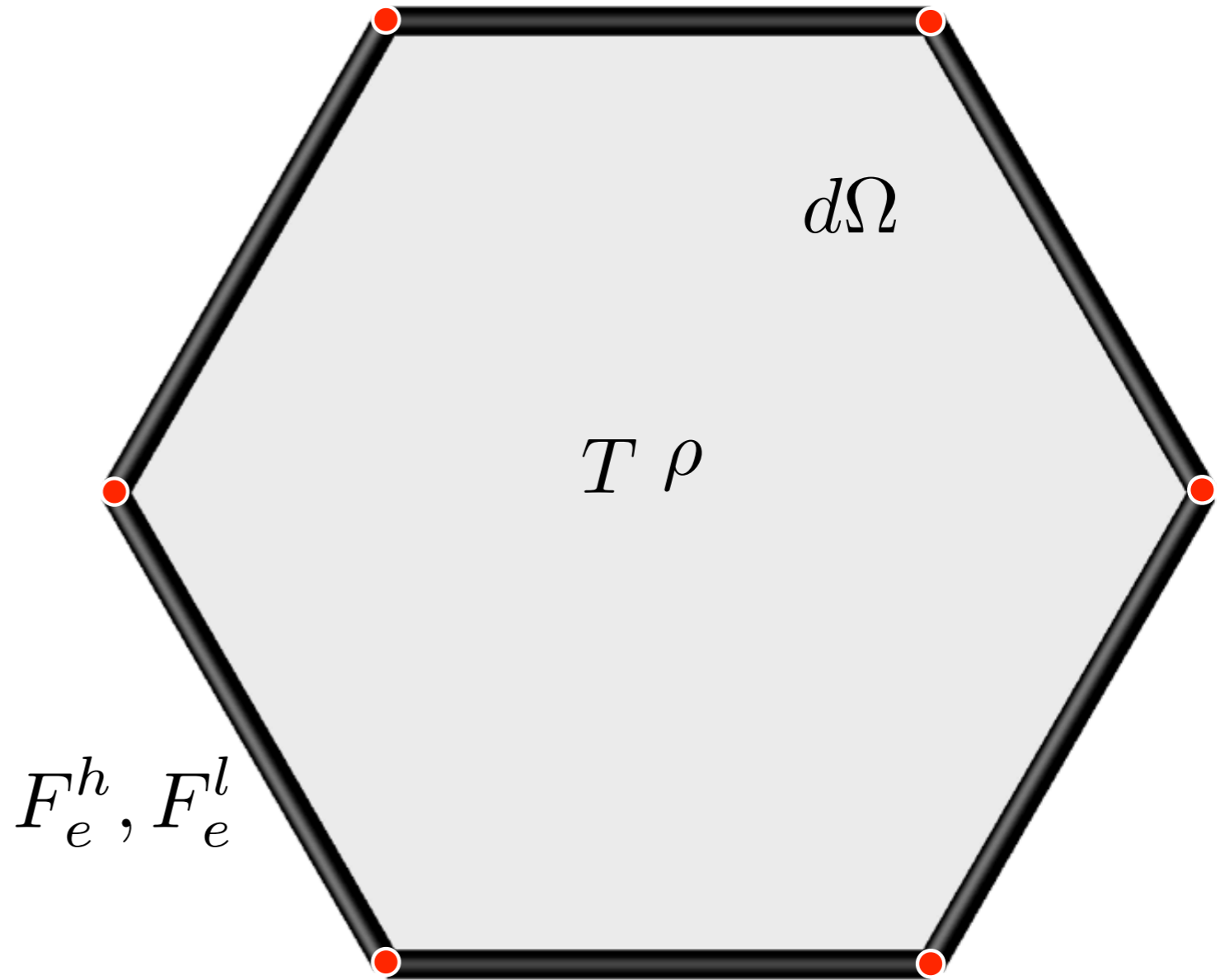
$$\int_{d\Omega} \nabla \cdot (\rho T \vec{u}) d\Omega = \oint_{dl} \rho T \vec{u} \cdot d\vec{l} \approx \sum_{i=1}^N \hat{\rho}_e \hat{T}_e \vec{u}_e \cdot d\vec{l}_e = \sum_{i=1}^N F_e$$

With traditional flux-corrected transport methods, we have two flux estimates: a high-order estimate and a low-order estimate.

Both fluxes are conservative by construction. The low-order flux also guarantees monotonicity.

We attempt to take as much of the high-order flux as possible while not violating monotonicity.

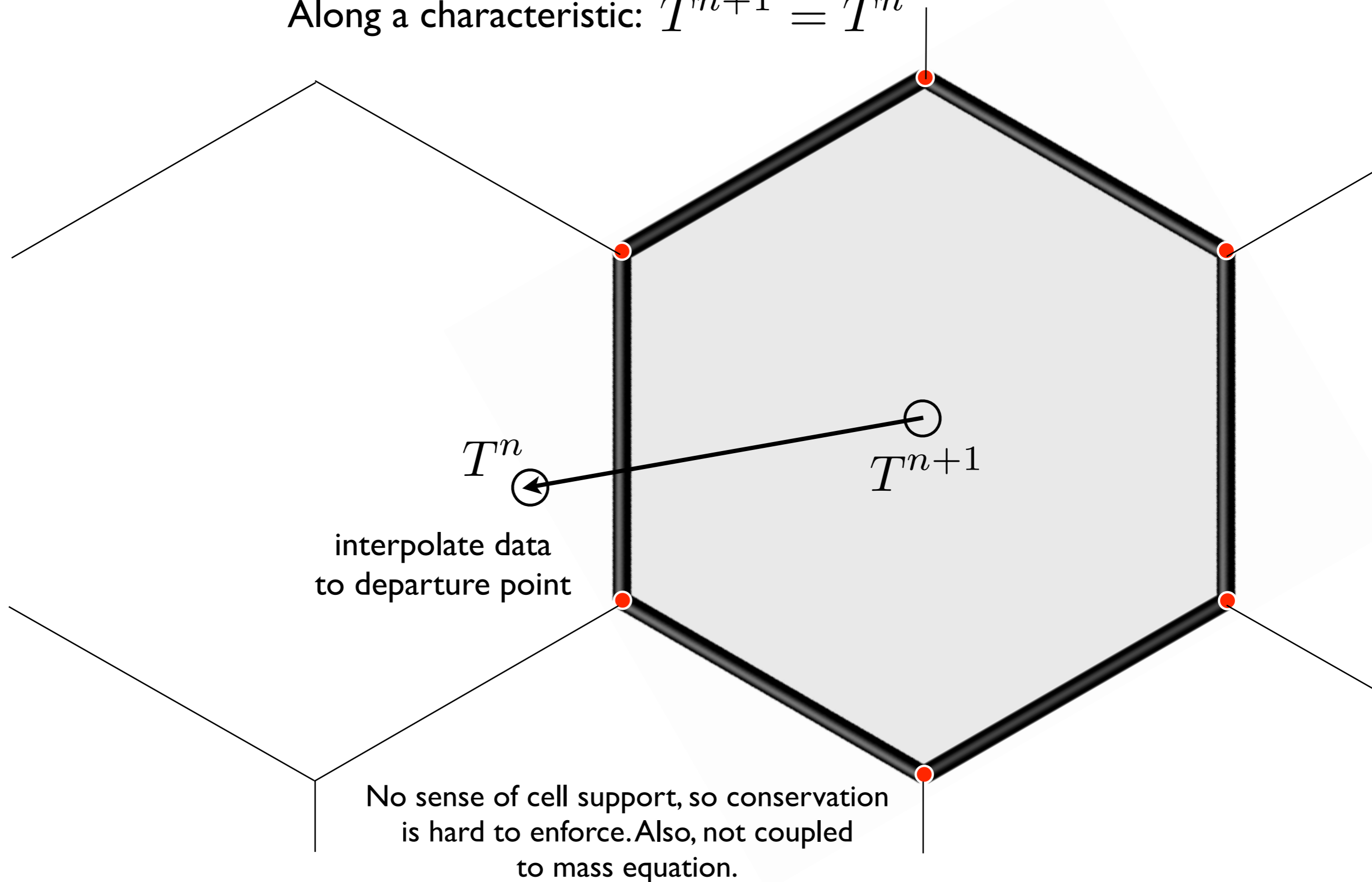
Relatively expensive and cost is linear with number of tracers.



$$F_e = \gamma F_e^h + (1 - \gamma) F_e^l$$

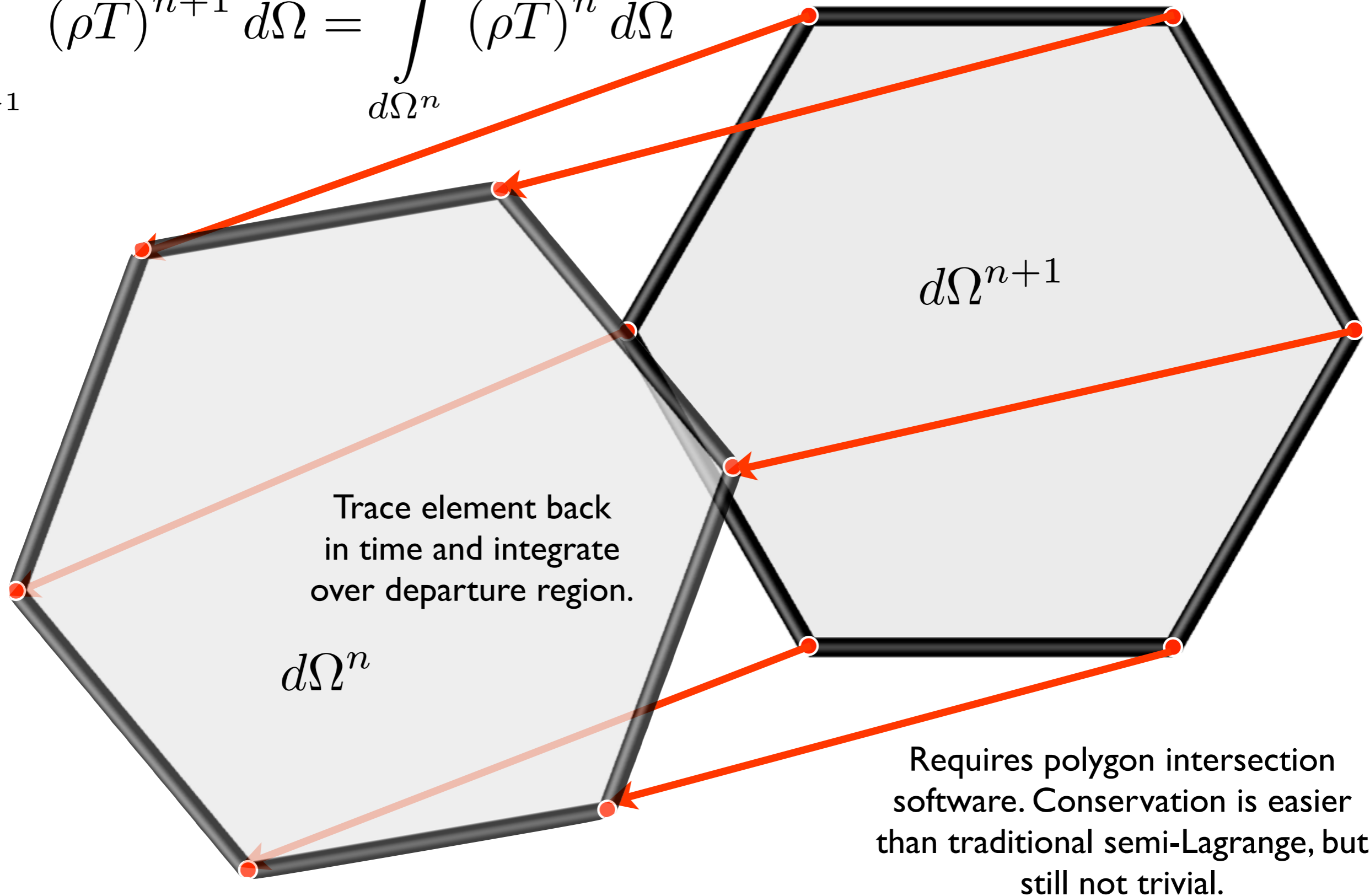
Alternative #2: Traditional Semi-Lagrange Transport

Along a characteristic: $T^{n+1} = T^n$

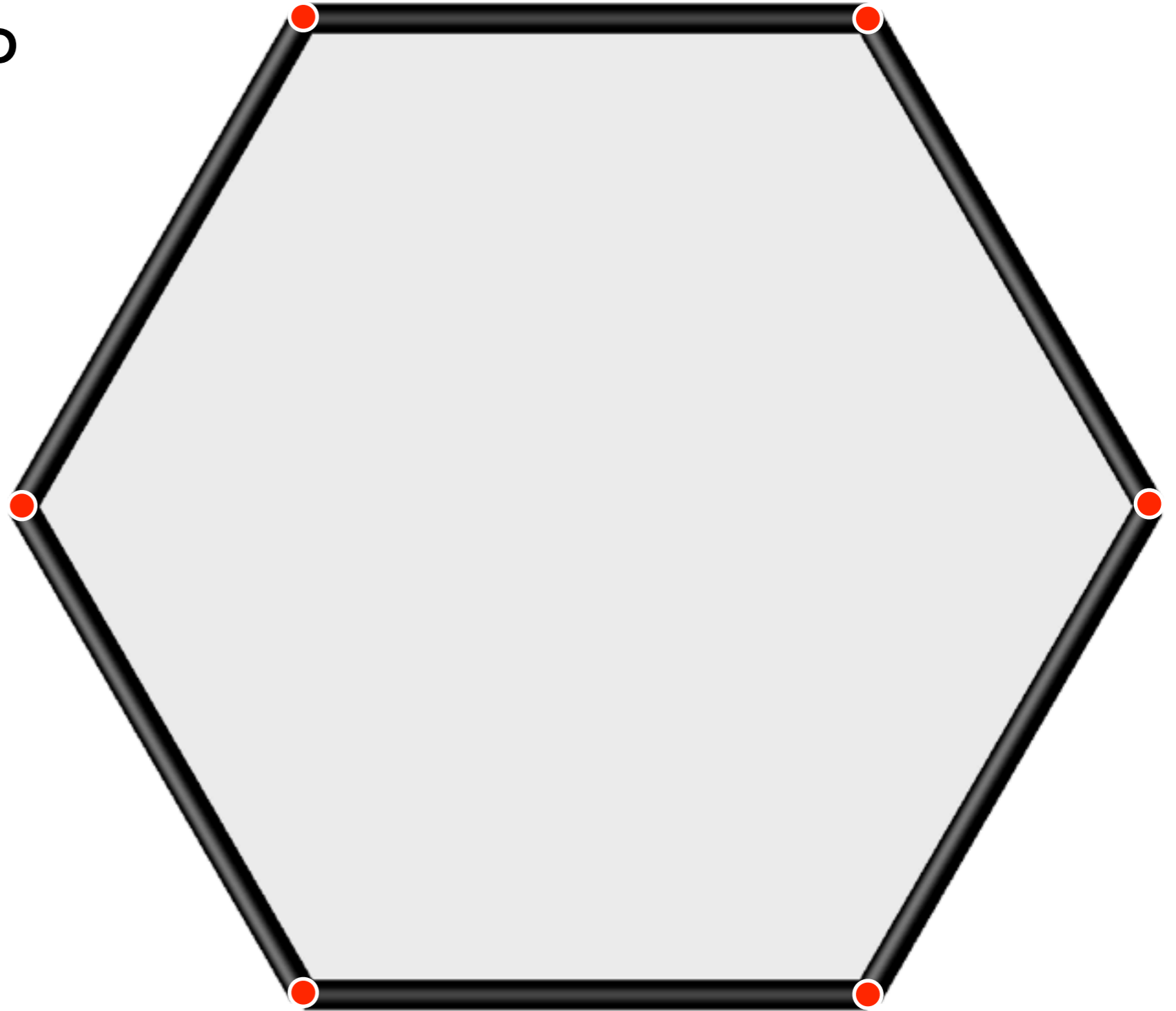


Alternative #2a: Cell-Integrated Semi-Lagrange Transport

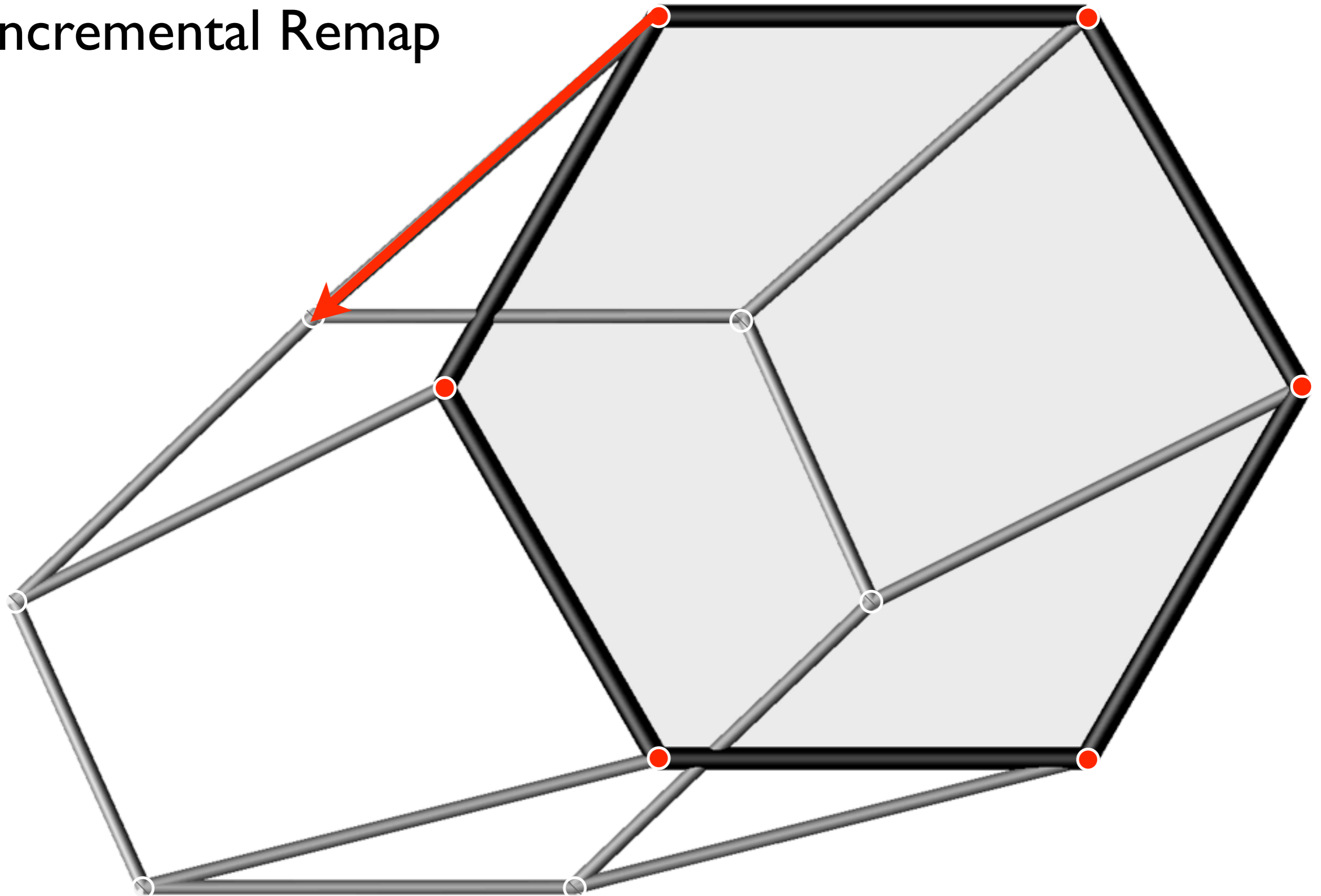
$$\int_{d\Omega^{n+1}} (\rho T)^{n+1} d\Omega = \int_{d\Omega^n} (\rho T)^n d\Omega$$



Alternative 3: Incremental Remap

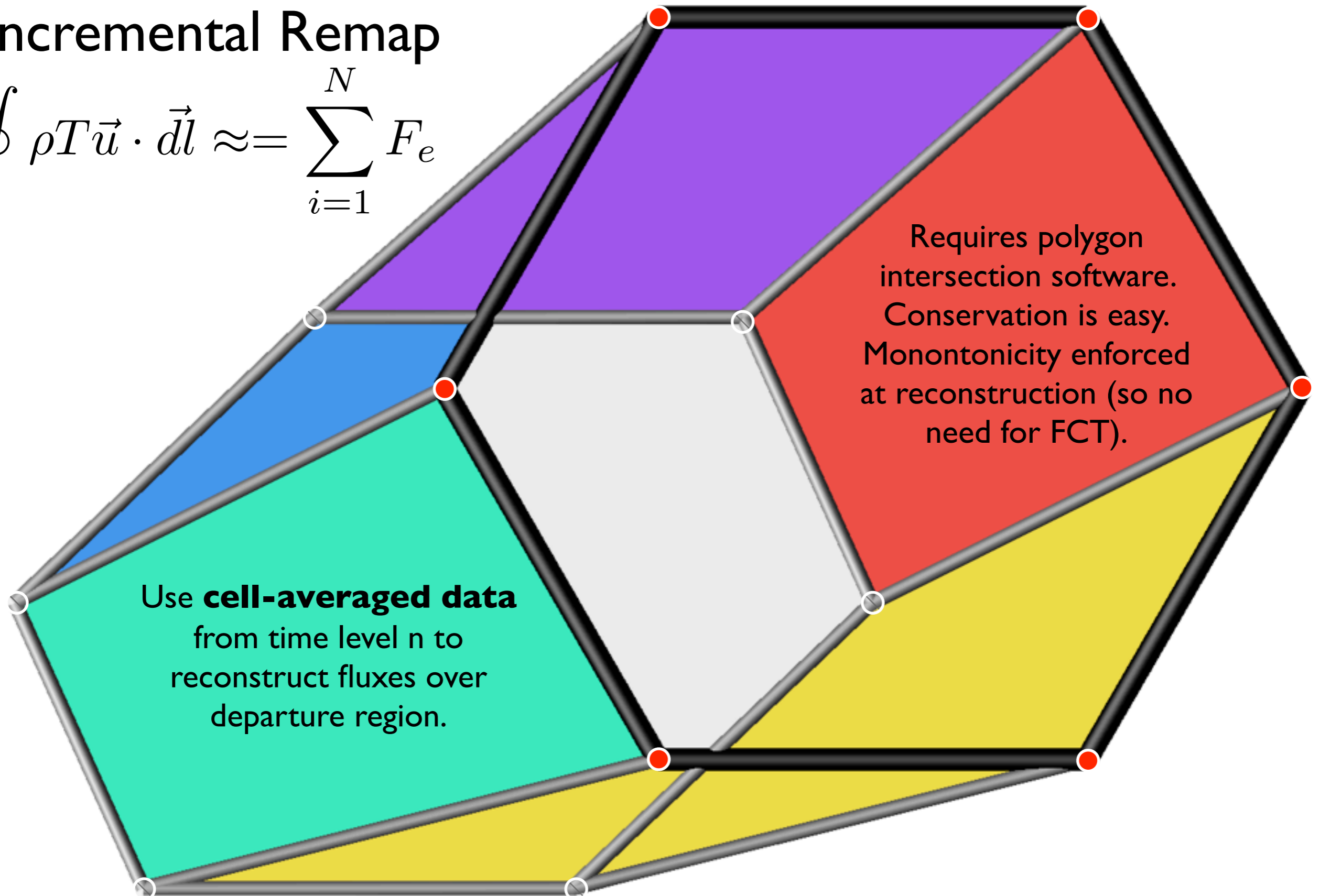


Alternative 3: Incremental Remap

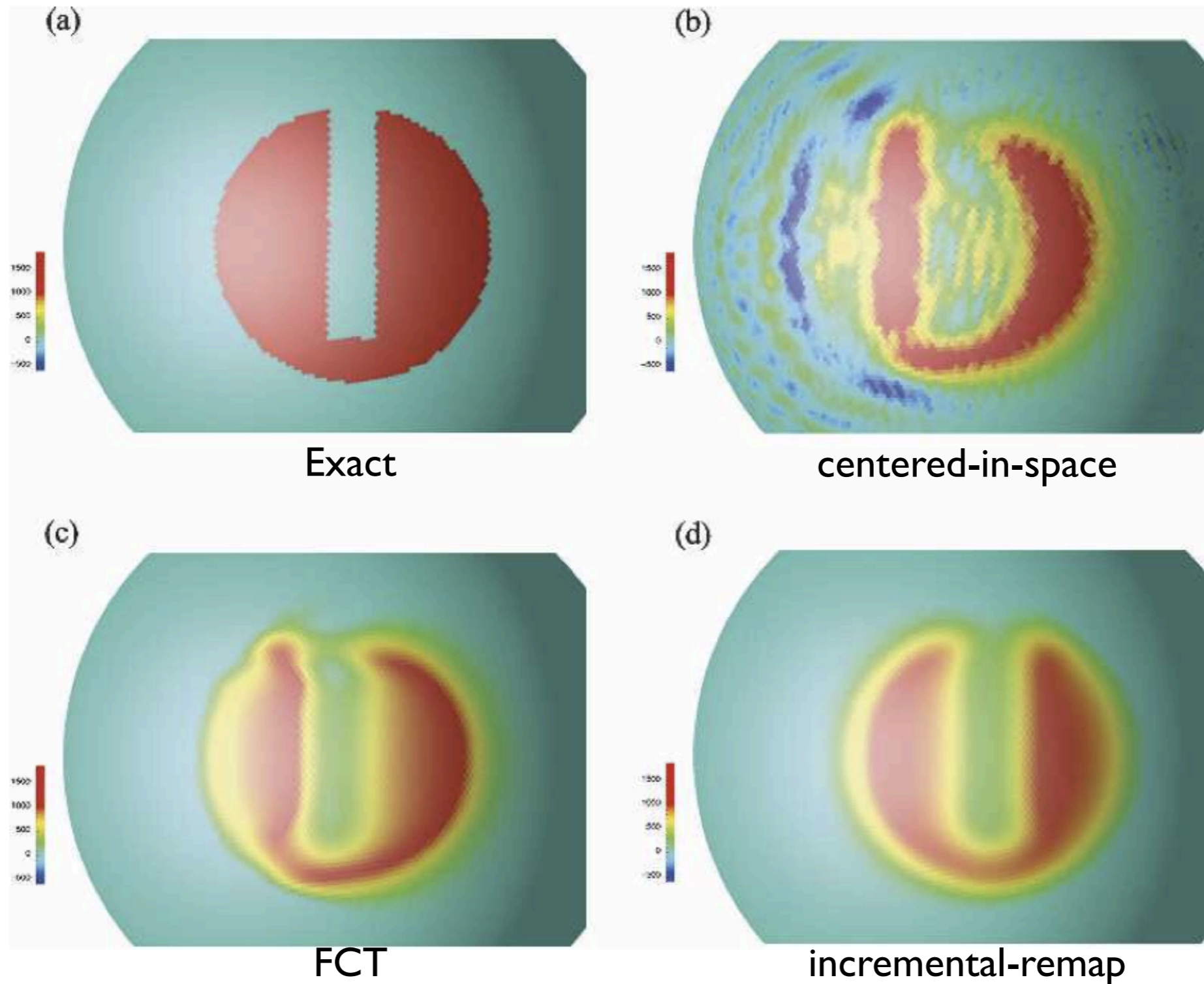


Alternative 3: Incremental Remap

$$\oint_{dl} \rho T \vec{u} \cdot d\vec{l} \approx \sum_{i=1}^N F_e$$



Incremental remap compares well to FCT.



- Lipscomb, W.H. and T. Ringler, 2005: An Incremental Remapping Transport Scheme on a Spherical Geodesic Grid, Monthly Weather Review, 133, 2335-2350. doi10.1175/MWR2983.1([link](#))

Our preferred alternative: Incremental Remap at High-Order

Assume a functional structure of the tracer within each convex polygon:

$$T(\vec{x}, t^n) = \sum_{j=1}^N c_{k,j}^n \beta_{k,j}(\vec{x})$$

With, for example, a basis of the form:

$$\beta_{k,j}(\vec{x}) \in \{1, x, y, x^2, xy, y^2\}$$

Need a method for updating basis coefficients:

$$c_{k,j}^n$$

Developing the Incremental Remapping at High Order (IRHO) method:

Given our tracer transport equation:

$$\partial_t(\rho T) + \nabla \cdot (\rho T \vec{u}) = 0$$

Multiply through by a test function and rearrange:

$$\partial_t(\phi_{k,i} \rho T) + \nabla \cdot (\phi_{k,i} \rho T \vec{u}) = \rho T \frac{D\phi_{k,i}}{Dt}$$

Integrate over the space-time element $\partial\Omega_{k,f} \times [t^n, t^{n+1}]$

$$\int_{\Omega_k} [(\phi_{k,i} \rho T)^{n+1} - (\phi_{k,i} \rho T)^n] d\Omega + \int_{t^n}^{t^{n+1}} \oint_{\partial\Omega_k} \phi_{k,i} \rho T \vec{u} \cdot \vec{n} ds dt = \int_{t^n}^{t^{n+1}} \int_{\Omega_k} \rho T \frac{D\phi_{k,i}}{Dt} d\Omega dt$$

Breaking the method into two systems:

Instead of solving this:

$$\partial_t(\phi_{k,i}\rho T) + \nabla \cdot (\phi_{k,i}\rho T \vec{u}) = \rho T \frac{D\phi_{k,i}}{Dt} d\Omega dt$$

We choose this:

$$\begin{aligned}\partial_t(\phi_{k,i}\rho T) + \nabla \cdot (\phi_{k,i}\rho T \vec{u}) &= 0 \\ \frac{D\phi_{k,i}}{Dt} &= 0\end{aligned}$$

Note that the original incremental remap scheme is included here.

Solving Part I of the IRHO system:

$$T(\vec{x}, t^n) = \sum_{j=1}^N c_{k,j}^n \beta_{k,j}(\vec{x})$$

$$\int_{\Omega_k} [(\phi_{k,i} \rho T)^{n+1} - (\phi_{k,i} \rho T)^n] d\Omega + \sum_f \int_{\Omega'_{k,f}} (\phi_{k,i} \rho T)^n d\Omega = 0$$

flux is computed by integrating over swept region

$$\Omega'_{k,f} = \partial\Omega_{k,f} \times [t^n, t^{n+1}]$$

Still need a method for determining how to evaluate the test function:

$$\phi_{k,i}(\vec{x}, t)$$

Solving Part II of the IRHO system:
Test function evaluation must satisfy:

$$\frac{D\phi_{k,i}}{Dt} = 0$$

recall

$$T(\vec{x}, t^n) = \sum_{j=1}^N c_{k,j}^n \beta_{k,j}(\vec{x})$$

We satisfy this requirement with:

$$\phi_{k,i}(\vec{x}, t) = \beta_{k,i}(\vec{\Gamma}(\vec{x}, t))$$

$$\vec{\Gamma}(\vec{x}, t) = \vec{x} + \int_t^{t^{n+1}} \vec{u}(\vec{\Gamma}(\vec{x}, \xi), \xi) d\xi$$

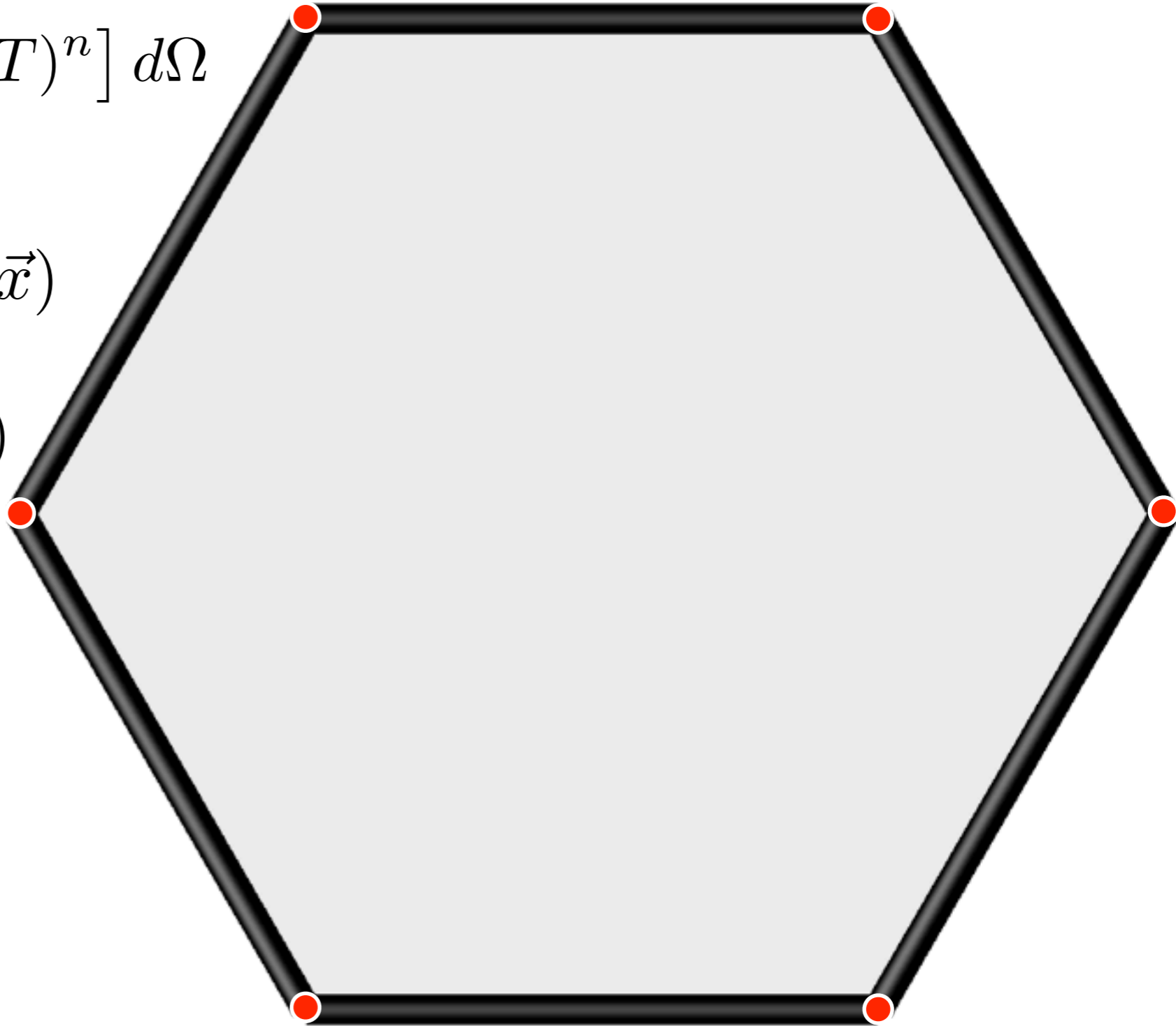
$$\vec{\Gamma}(\vec{x}, t) = \vec{x} + (t^{n+1} - t)\vec{u}$$

Cell integration

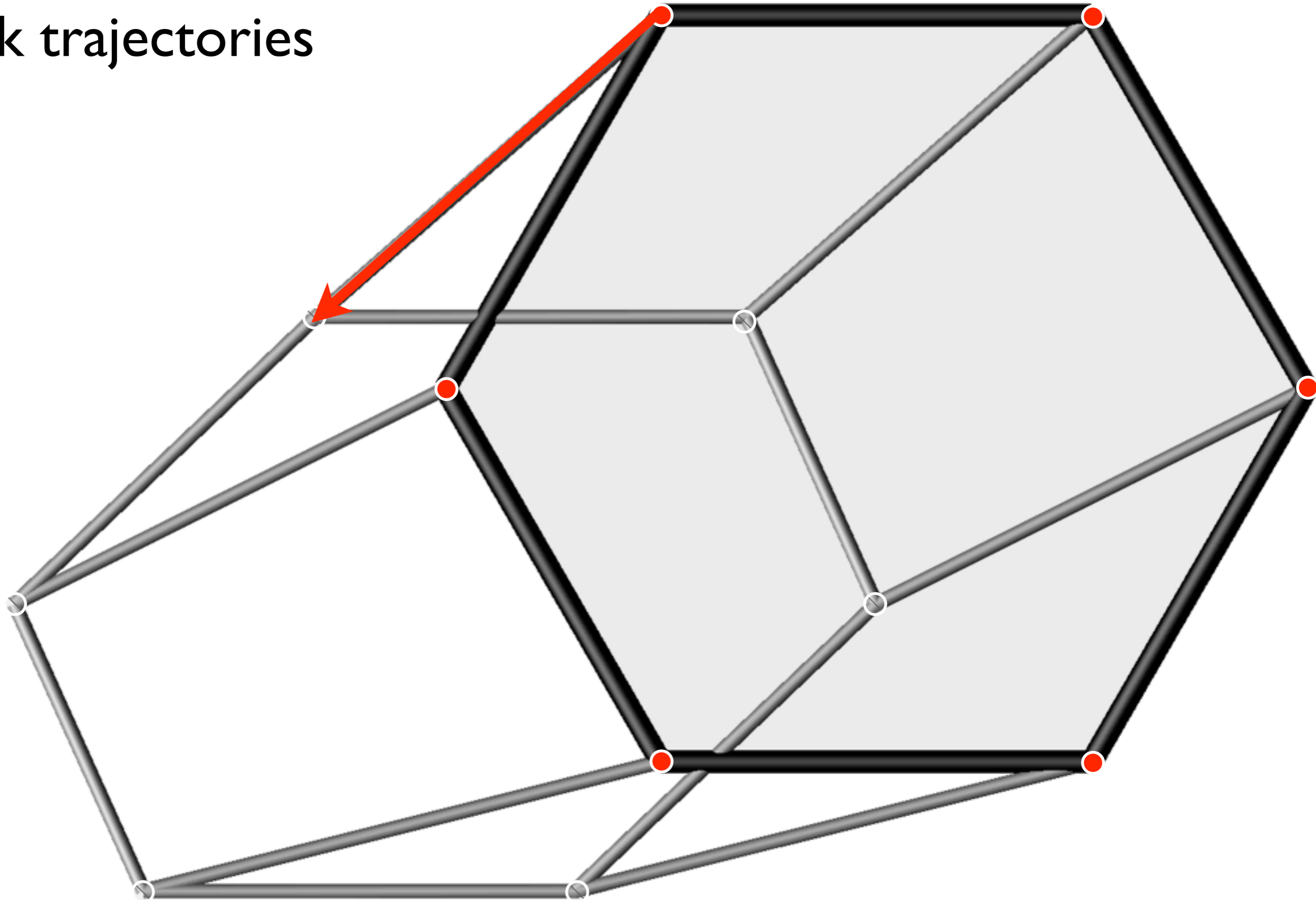
$$\int_{\Omega_k} [(\phi_{k,i} \rho T)^{n+1} - (\phi_{k,i} \rho T)^n] d\Omega$$

$$T(\vec{x}, t^n) = \sum_{j=1}^N c_{k,j}^n \beta_{k,j}(\vec{x})$$

$$\phi_{k,i}(\vec{x}, t) = \beta_{k,i}(\vec{\Gamma}(\vec{x}, t))$$

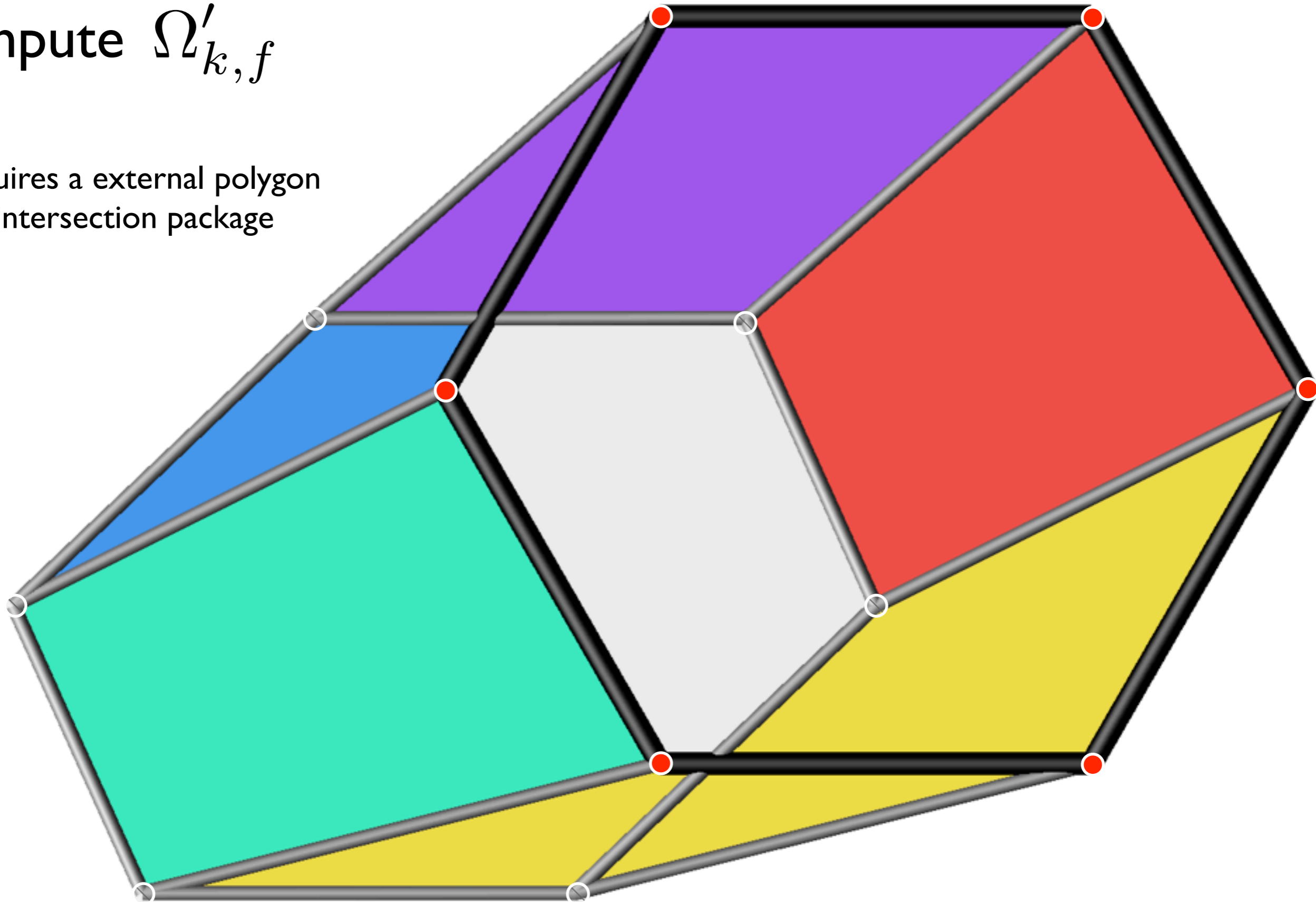


Flux calculation: back trajectories



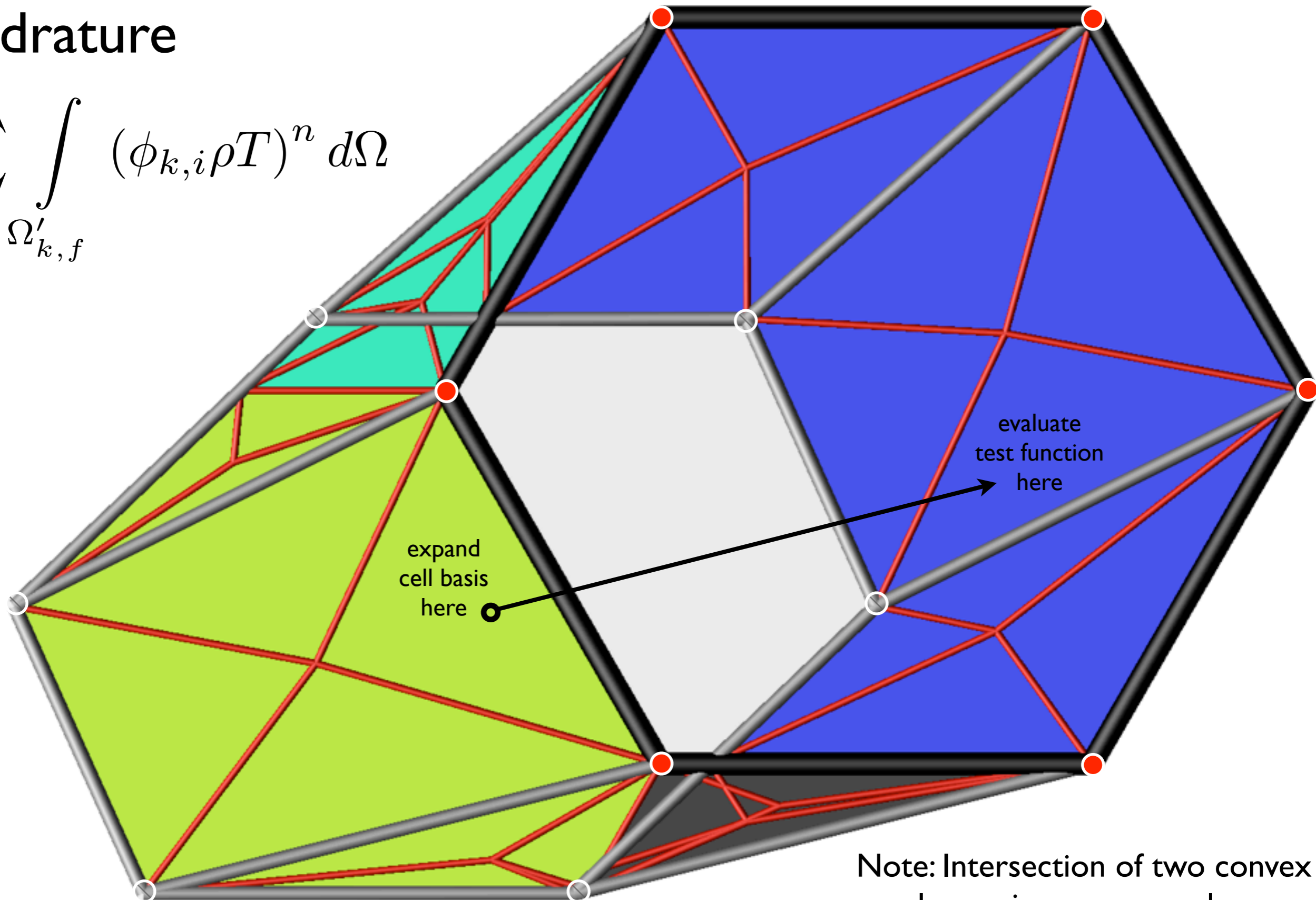
Flux calculation:
compute $\Omega'_{k,f}$

requires a external polygon
intersection package



Flux calculation: quadrature

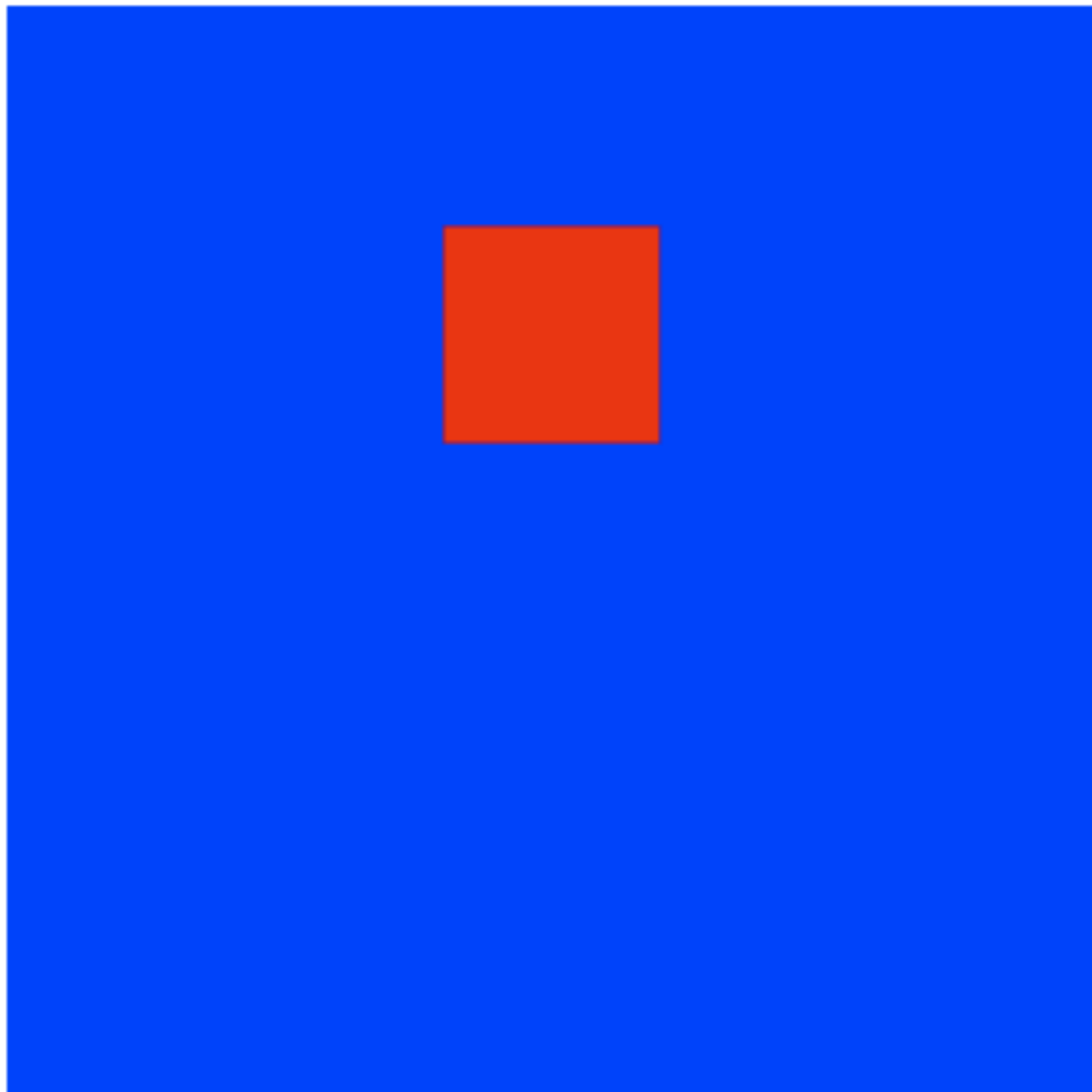
$$\sum_f \int_{\Omega'_{k,f}} (\phi_{k,i} \rho T)^n d\Omega$$



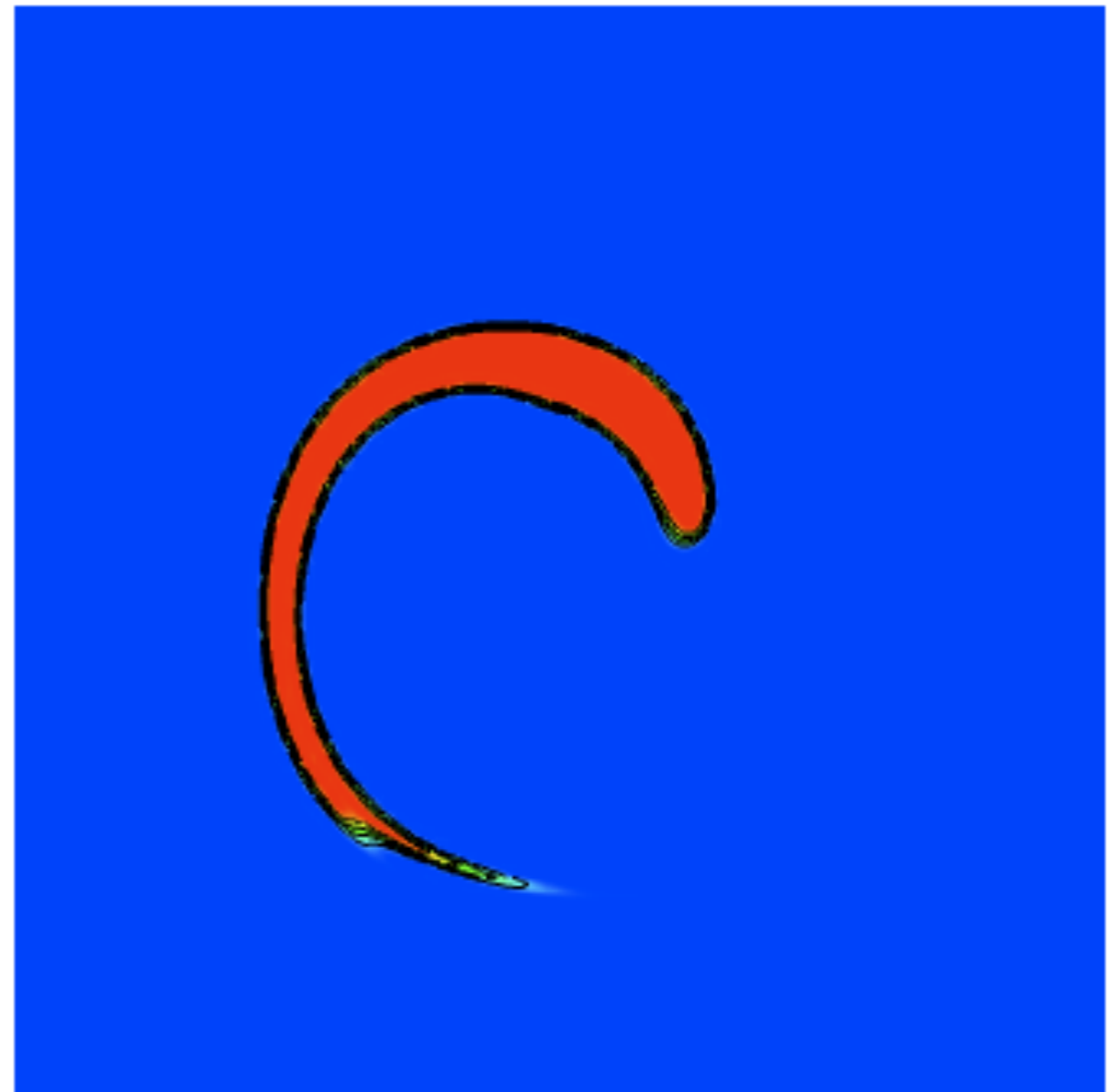
Note: Intersection of two convex polygons is a convex polygon.

Deformation of a Square (period=4): Exact solution

$t = 0$ and $t = 4$

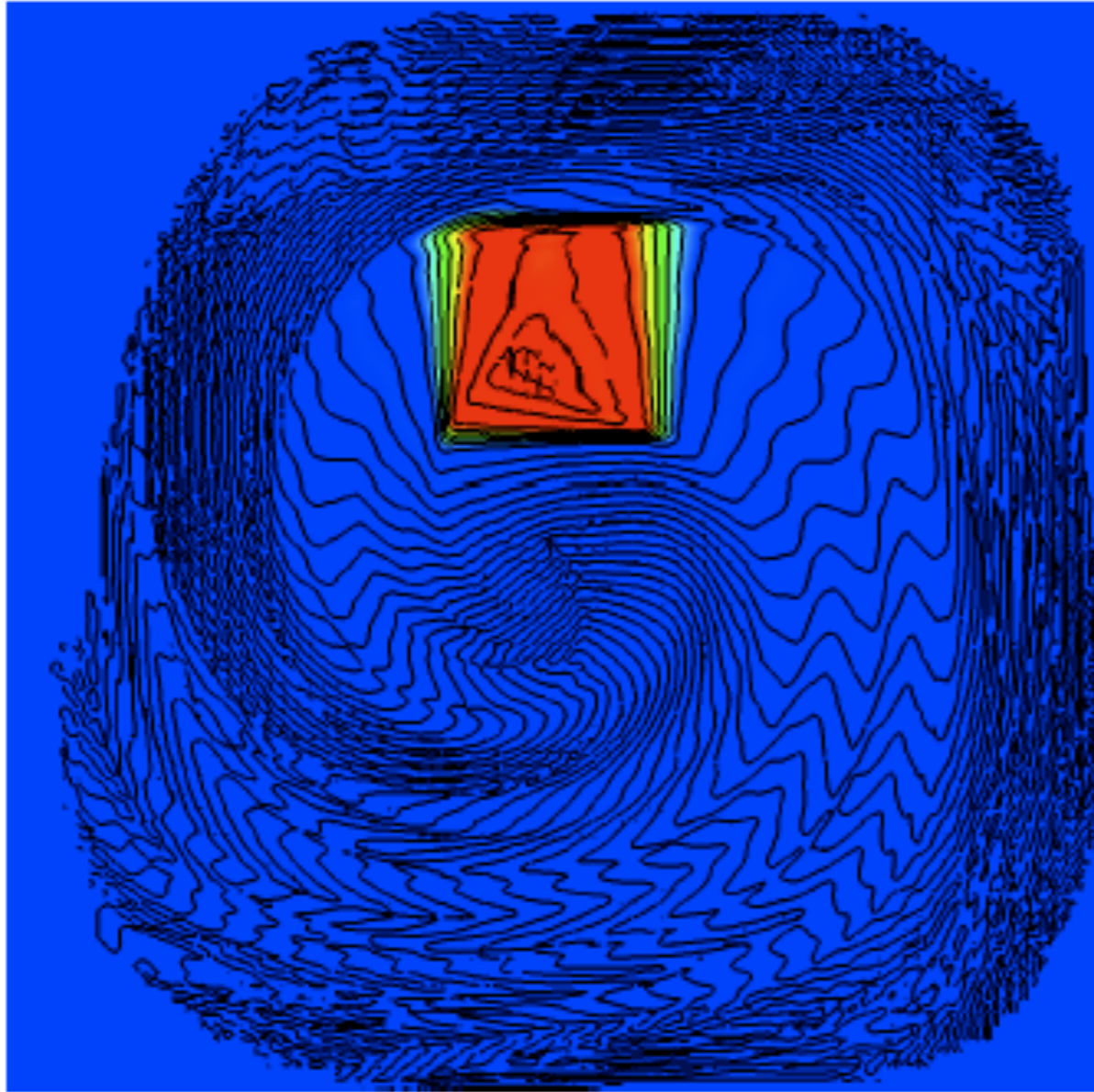


$t = 2$

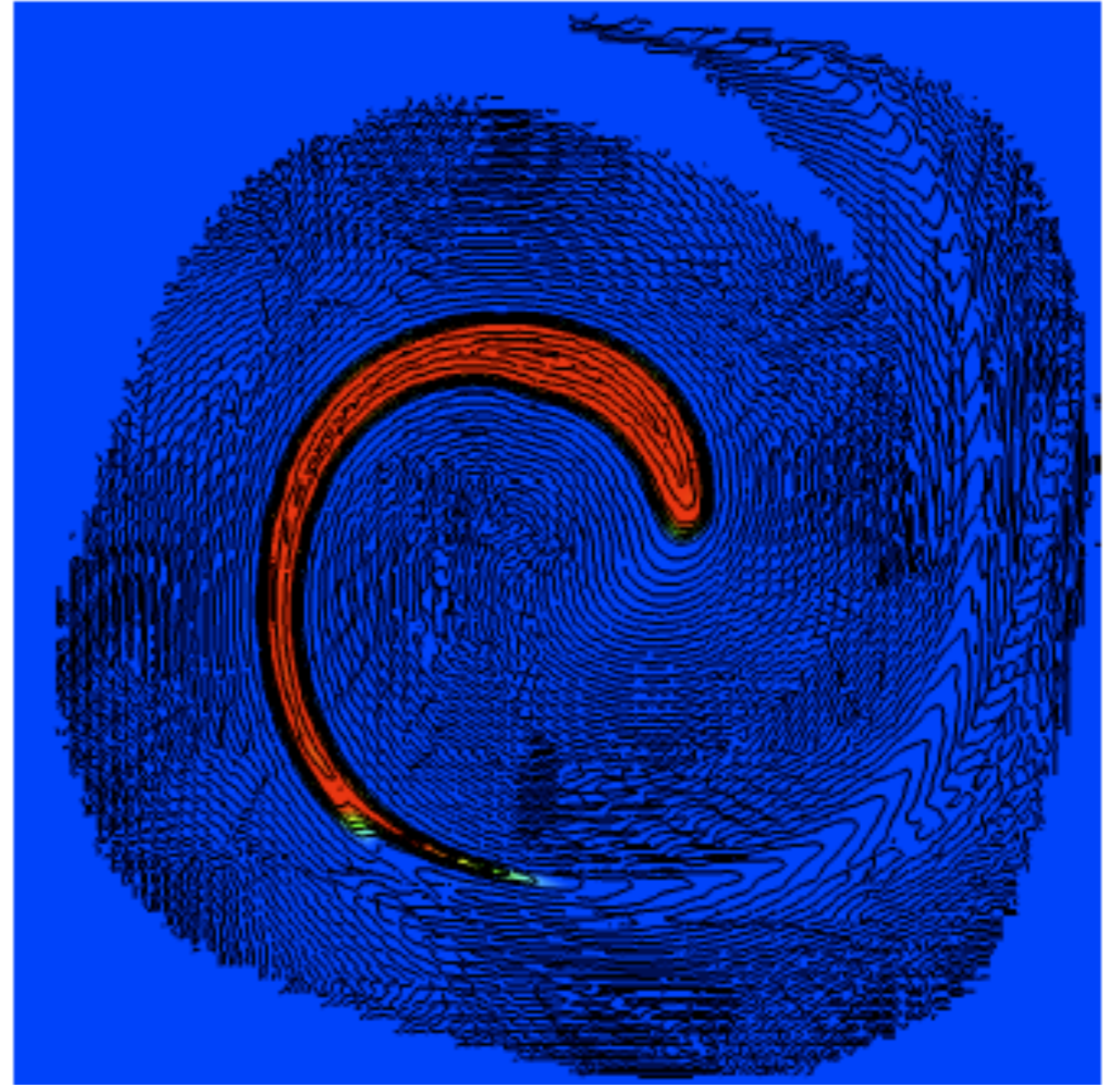


Deformation of a Square (period=4): Tracer solution with without limiting (on a quad mesh)

$t = 4$

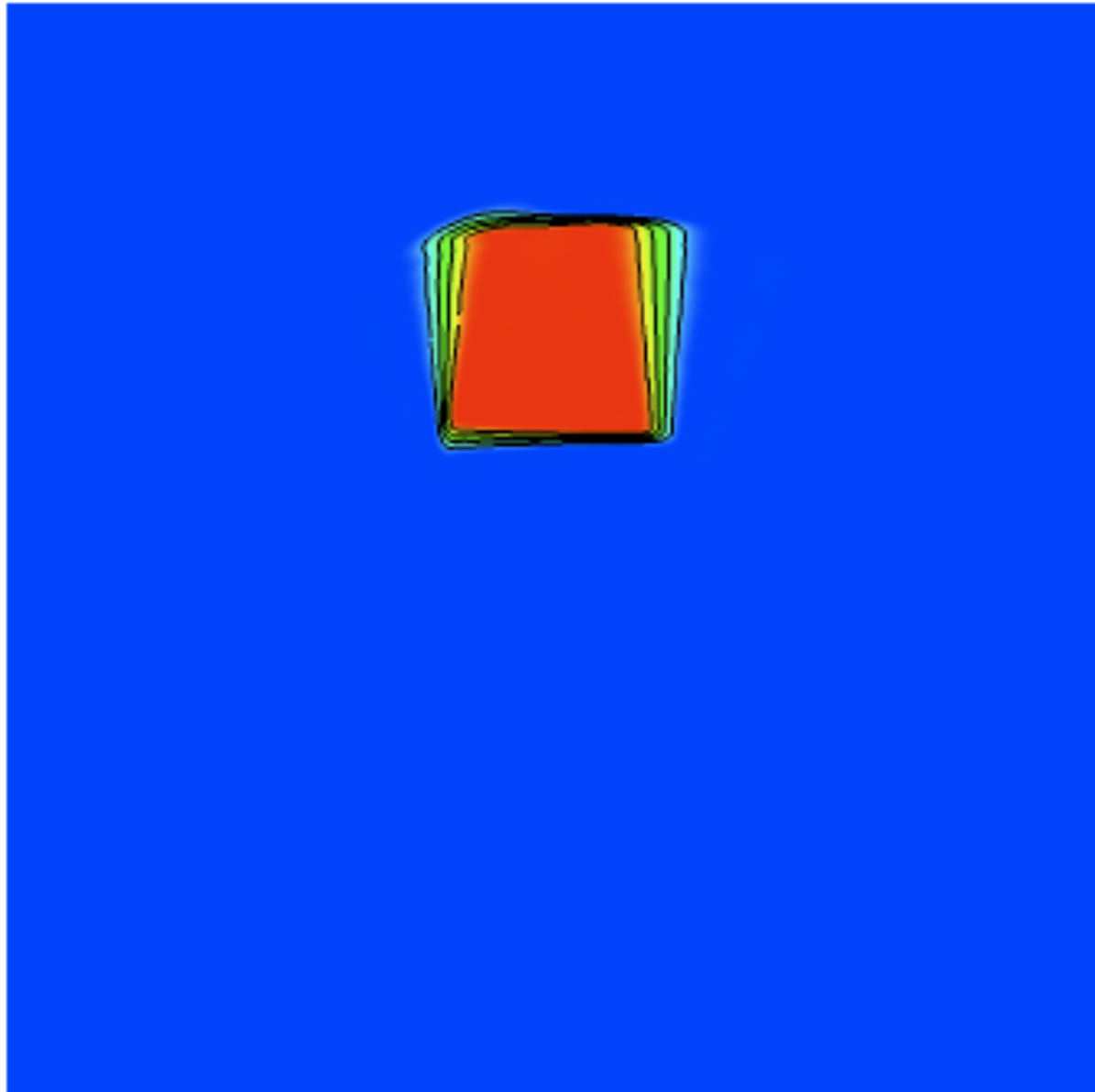


$t = 2$

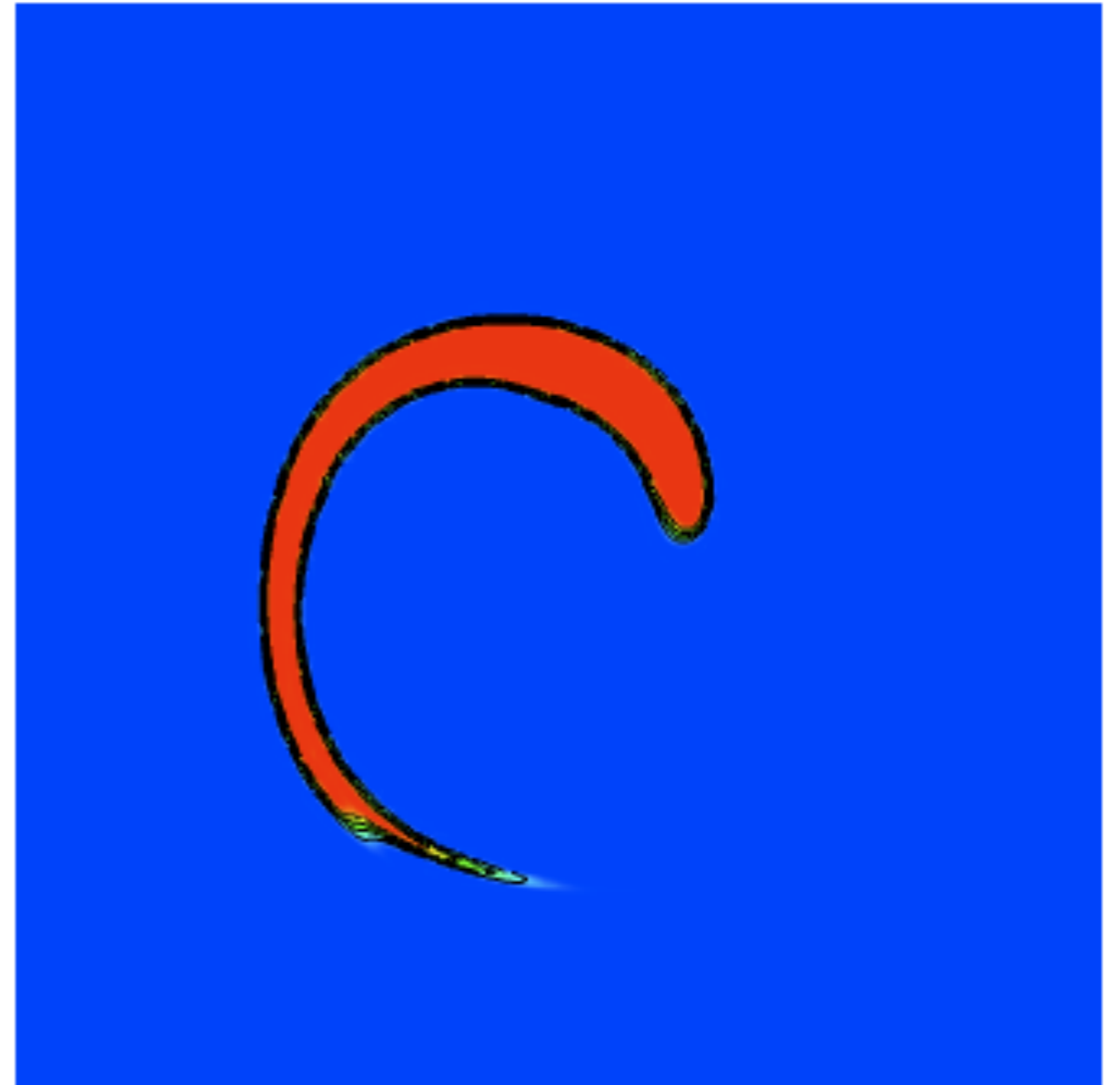


Deformation of a Square (period=4): Tracer solution with limiting

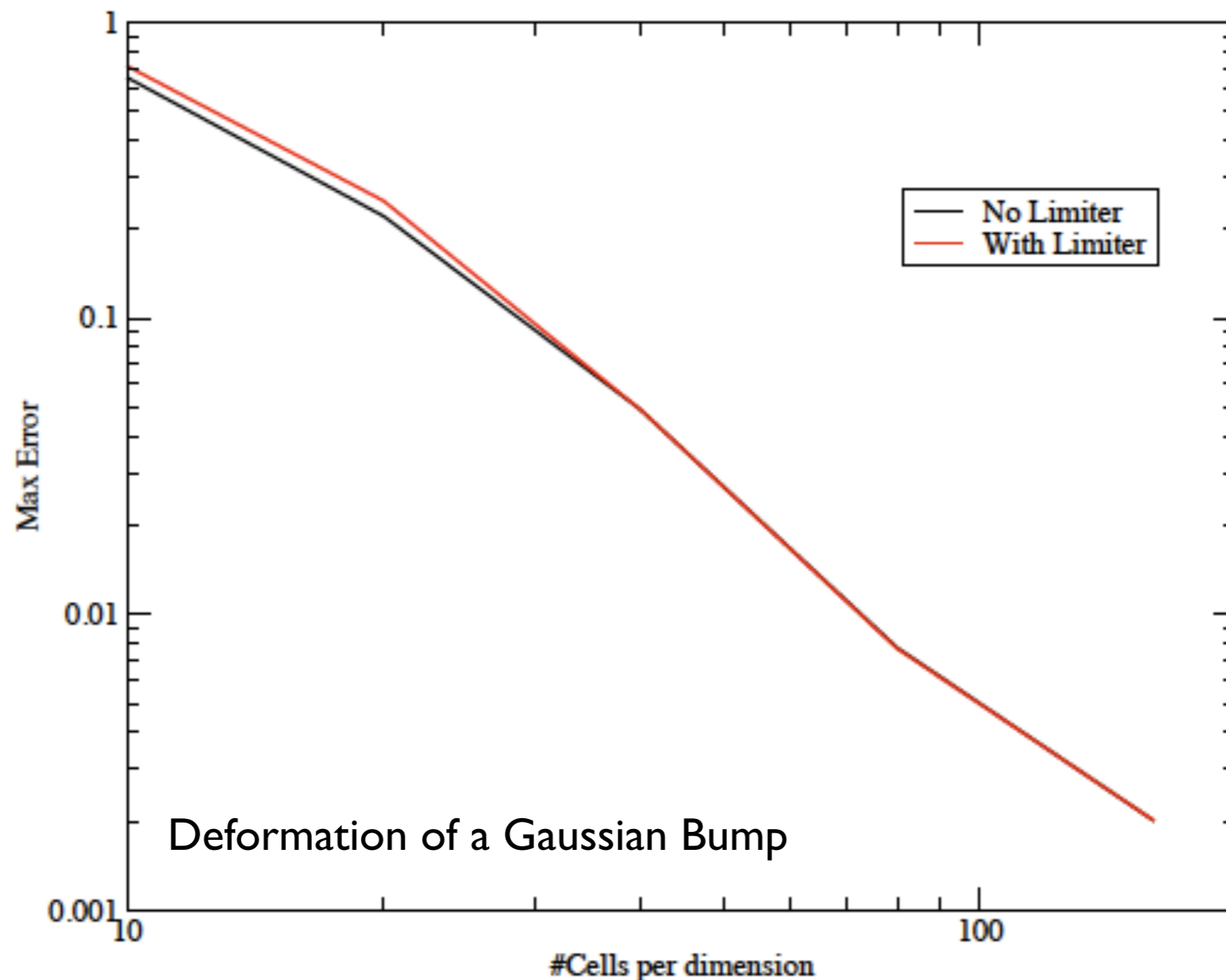
$t = 4$



$t = 2$

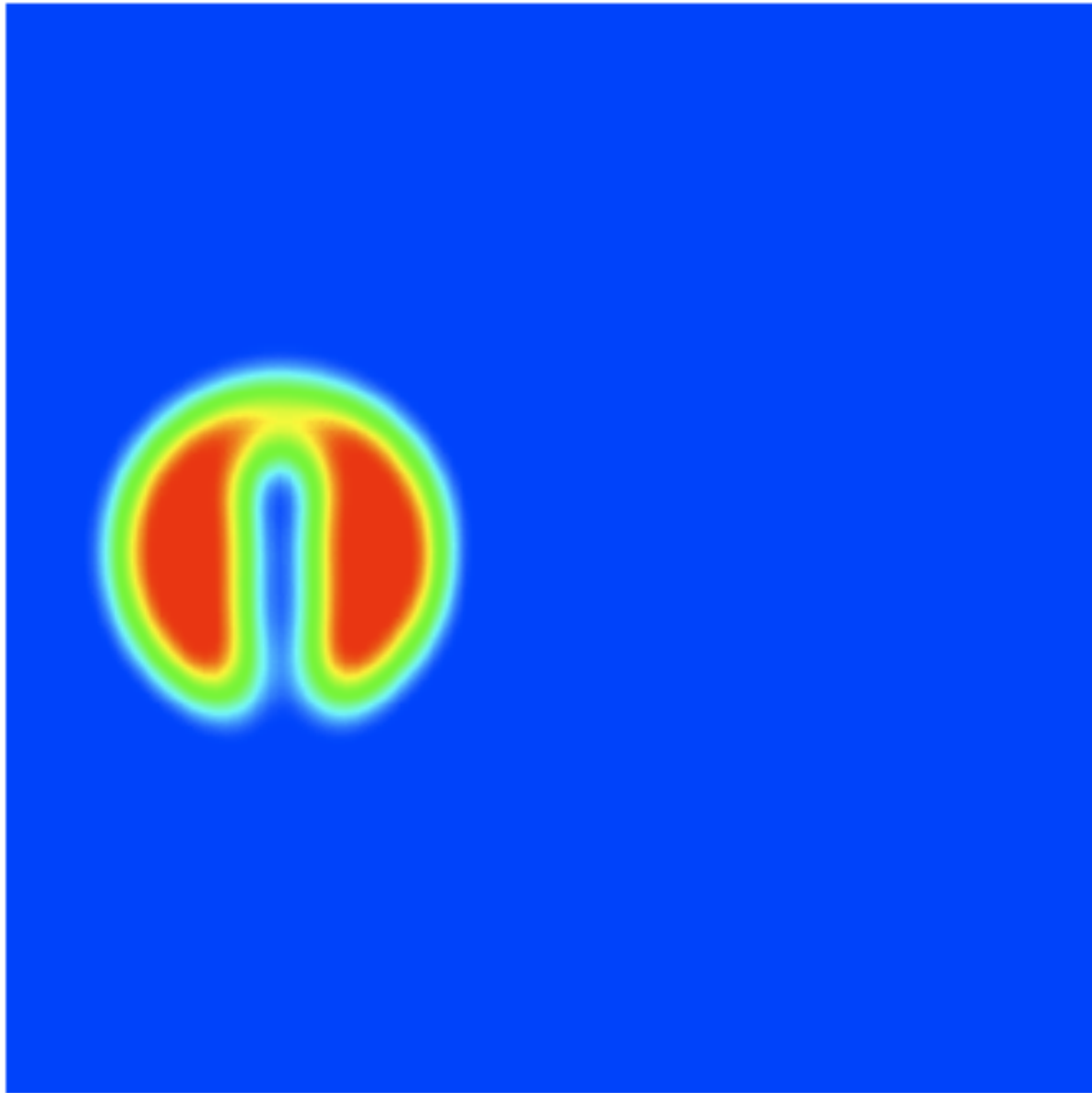


I have not talked about the limiting of the flux calculation, but it is critical to the method. Importantly, the limiting does not reduce accuracy when the solution is smooth.

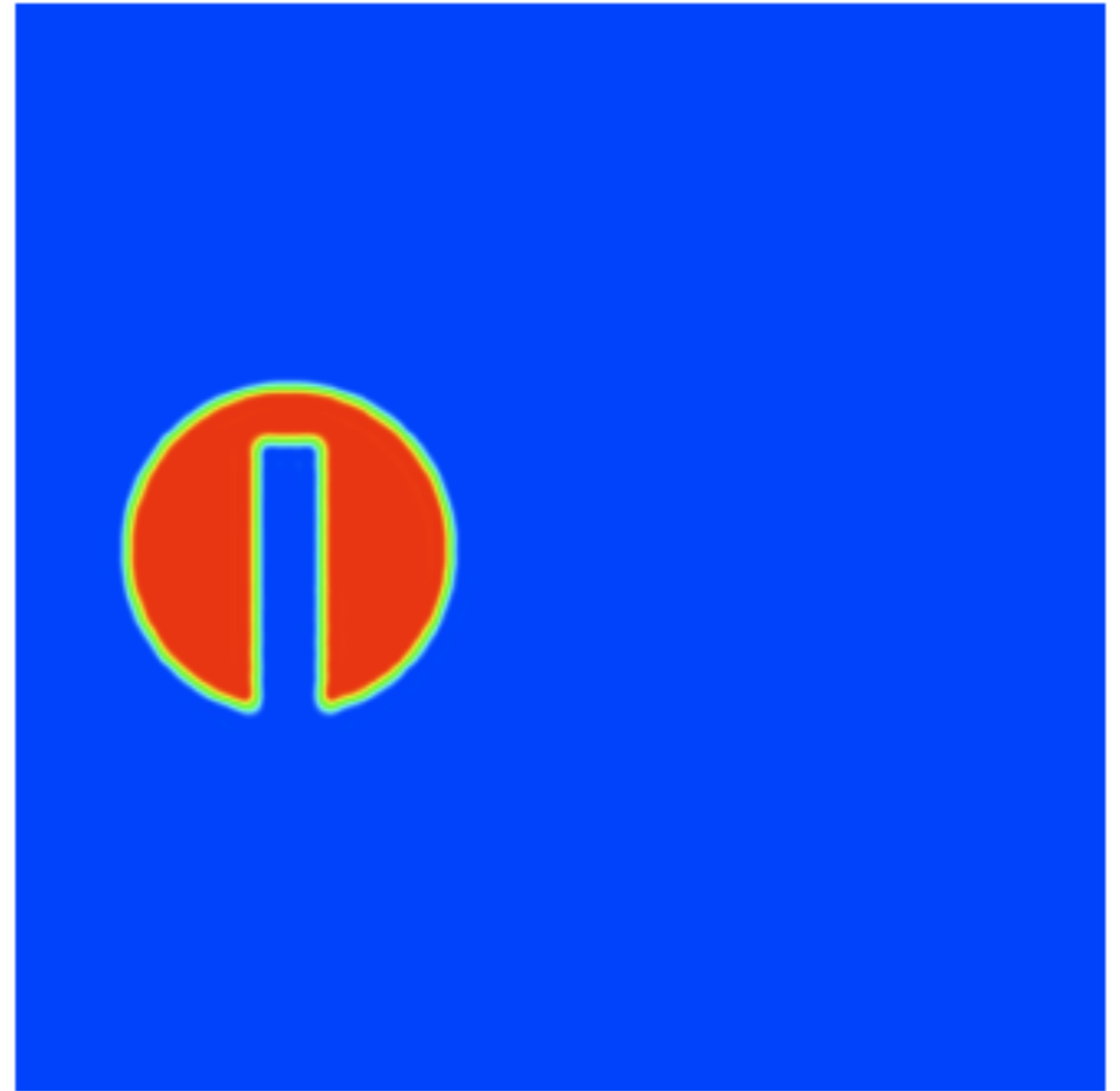


Solid Body Rotation: One revolution of slotted cylinder

CDG(1)



CDG(3)



In closing, let us recall the attributes we seek:

1. Locally conservative (and, thus, globally conservative) for some domain Ω .

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Thoughts on Incremental Remapping at High-Order

1. Conservative by construction
2. Arbitrary order of accuracy on convex polygons
3. Accommodates large CFL by construction
4. We know where each part of the tracer flux departs and arrives, so the limiting is precise.
5. Expensive to do the first tracer!



Thanks!

Alternative #3: Flux-Form Semi-Lagrange (aka Incremental Remap)

